GROUP COVERS WITH A SPECIFIED PAIRWISE INTERSECTION

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ABSTRACT

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A collection of subgroups whose union is equal to the whole group is known as a group cover. Various types of group covers have been explored in recent years. In this work, we define a new type of cover, an equal L-intersection cover, which is specified by the size and pairwise intersection of the subgroups involved. We demonstrate that this cover is in fact distinct from previous classes of group covers. Further, we give tests to determine whether or not a finite group has this type of cover. These tests are implemented in the computational algebra program `GAP` to find new instances of equal L-intersection covers. These examples lead to interesting questions and conjectures.

Chapter 1

Background on Groups

1.1 Basic Examples and Applications of Groups

Human beings have long been fascinated by symmetry. In mathematics, *groups* are algebraic structures which describe the symmetry of physical and mathematical objects. Though group theory can reach dizzying heights of abstraction, the basic ideas behind it are easy to understand. Before delving into the mathematically rigorous view of the subject, it will be nice to have an informal overview of what groups are and how they can be used.

As an example, imagine a square with a label on each corner. Consider the different ways in which this shape can be repositioned without changing its location. We can keep track of the state of the square by recording the relative positions of its labeled corners. First, we should note that technically the figure could be rotated by 0 degrees, leaving it exactly as it started. It can also be rotated by 90, 180, or 270 degrees. In addition to these rotations, the square has four axes of symmetry–horizontal, vertical, and two diagonal–over which it can be flipped. A little thought reveals that for any sequence of rotations or reflections, there is another motion which would directly achieve the same effect. Obviously, a rotation of 90 degrees followed by a degree of 180 degrees is equivalent to a rotation of 270 degrees, while two rotations of 180 degrees yields the same effect no rotation. A horizontal flip followed by a vertical flip yields the same state as a rotation by 180 degrees. Two vertical flips return the square to its starting point, leaving it unchanged.

These eight motions (four rotations and four reflections) form the symmetry group of the square. Note that any motion can be undone, i.e. the square can always be returned to its starting point. This is one of the defining notions of a group. This group is known as the *dihedral group of order 8*, abbreviated D_4 since it acts on a figure with 4 vertices. We will use this group in several examples.

In addition to D_4, familiar examples of groups include the integers with the operation of addition and the nonzero real numbers with the operation of multiplication [**?**, **?**]. Another important class of groups are the groups of *addition modulo n* for some integer n, denoted $\mathbb{Z}_n$. These groups consist of the set $\{0, 1, \ldots, n-1\}$, with the sum of two numbers reduced to its remainder when divided by n. These groups are widely used today to assure the secure and efficient transmission of digital information [**?**].

In addition to codes, group theory has been used to approach many important mathematical problems. Mathematicians searched for centuries for a general approach to solving polynomial equations before the ideas behind group theory helped show that such a solution method is impossible for arbitrary polynomials of degree five or higher [**?**]. The use of groups is also commonplace in the physical sciences. Chemists use group theory to describe the structures of molecules and compounds [**?**]. An understanding of groups is considered critical to understanding the elementary particles underlying the structure of the universe [**?**]. Physicist Steven Weinberg, a recipient of the Nobel Prize, has expressed the belief that the universe is essentially composed of symmetry groups [**?**]. Groups also arise in some surprising and amusing ways related to less scientific pursuits. Several of M.C. Escher's artistic optical illusions can be analyzed via group-theoretic methods [**?**]. Group theory is at the core of the solution to popular puzzles such as the Rubik's Cube and Lights Out [**?**]. The subject can also be used to examine "perfect shuffles," a technique used on decks of cards by magicians and dishonest card-players [**?**].

1.2 Mathematical Preliminaries

We will now review some basic terminology and concepts in group theory. The material in this section is standard, and references can be found in Gallian[**?**] or Dummit and Foote

[?]. We assume that the reader is familiar with basic notation and terminology from set theory.

Formally, a group is a non-empty set G with a binary operation $\circ : G \times G \to G$ which satisfies the following requirements

1. *Identity.* There exists an element e such that $e \circ x = x \circ e = x$ for all $x \in G$. i.e. there is an element which leaves every group element fixed.

2. *Inverses.* Each element x has an inverse x^{-1} such that $x \circ x^{-1} = e = x^{-1} \circ x$. i.e. each group member has a corresponding member with which it cancels, leaving the identity.

3. *Associativity.* For all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$ i.e. the grouping of parentheses doesn't change the result of the expression

A generic group operation $\circ$ is often called *multiplication*. Often, when the operation is understood, group theorists will dispense with $\circ$ and simply write xy to mean $x \circ y$.

The *order* of a group, denoted $|G|$, is the number of elements in the group. If G has finitely many elements, G is called a finite group. If the order of G is a prime power, i.e. equal to p^n for some prime p and some positive integer n, G is said to be a *p-group*. The order of an element is defined to be the minimum number of times the element must be multiplied by itself to obtain the identity, i.e. the least integer n such that $x^n = e$. If no such n exists, the element is said to have infinite order. Note that, contrary to the rules of more familiar examples, there is no requirement that the group be commutative i.e. there are groups where $xy \neq yx$. Groups in which all elements commute are called *abelian*, in honor of group theory pioneer Niels Abel.

A *generating set* for a group is a collection of elements such that every element in the group can be expressed as the product of powers of those elements. The size of a minimal generating set is called the *rank* of the group.

There are a few basic families of groups which are frequently encountered as basic examples. The previously-mentioned $\mathbb{Z}_n$ groups are an example of an important class of groups known as *cyclic groups*, which have a generating set consisting of a single element.

The cyclic group of order n is also denoted C_n. These groups are also known as C_n. The *dihedral group of order n*, also written D_n, corresponds to the symmetries of a regular n-gon in the plane. The *symmetric group of degree n*, written S_n, is the set of all permutations on a set of n points.

A *subgroup* is a subset of a group that also forms a group in its own right. For instance, the set of all even integers under addition form a subgroup of the integers under addition, and the set consisting only of rotations is a subgroup of D_4. The *alternating group of degree n*, also known as A_n, is the subgroup of S_n consisting of all elements which have even order. Every group is technically a subgroup of itself; a subgroup which is not equal to the whole group is called a *proper subgroup*.

Several results in abstract algebra relate properties of a group to its order. One is *Lagrange's Theorem*, which states that in a finite group, the order of a subgroup always divides the order of the group [**?**]. Another is *Cauchy's Theorem*, which states that if p is a prime which divides the order of a finite group, the group will have an element of order p. If there exists a positive integer n such that $g^n = e$ for all $g \in G$, we that say that G has *exponent n*. This is often denoted $exp(G) = n$ [**?**].

We can relate groups to one another via *homomorphisms*. If G_1 and G_2 are finite groups, a *homomorphism* ϕ is a map from G_1 to G_2 such that $\phi(xy) = \phi(x)\phi(y)$. In this case, we say G_1 is the *homomorphic image* of G_2. It is not hard to show that homomorphisms preserve subgroups, i.e. if H is a subgroup of G_1, $\phi(H)$ is a subgroup of G_2. If the homomorphism is a bijection, it is called an *isomorphism*. If there is an isomorphism from G_1 to G_2, we say that G_1 and G_2 are *isomorphic*. We may write this as $G_1 \cong G_2$.

Given two groups G_1 and G_2, it is possible to form a new group called the *direct product* of G_1 and G_2, denoted $G_1 \times G_2$, by considering all ordered pairs of the form (g_1, g_2), with $g_1 \in G_1$ and $g_2 \in G_2$, and operation defined componentwise. It is worthwhile to note that every finite Abelian group is isomorphic to the direct product of cyclic groups [**?**]. An *elementary abelian group* is one of order p^n which has exponent p for some prime p and some positive integer n. Such group are isomorphic to the direct product of n cyclic groups of order p.

1.3 Certain Types of Groups and Subgroups

Here we will remind the reader of certain definitions and results which will eventually be encountered in the remainder of the work. The exposition is intended to be cursory, and at times we lean heavily on the textbook by Dummit and Foote [**?**].

Let G be a group. The *center* of G, denoted by $Z(G)$, is defined to be the set of all elements which commute with every element in the group, i.e. a subset of the form

$$Z(G) = \{x \in G | xg = gx \forall x \in G\}.$$

If G is a group and H is a subgroup of G, a subsets of the form

$$aH = \{ah | a \in G, h \in H\}$$

is called a *left coset of H.* Similarly, the subsets of the form Ha are called *right cosets.* A subgroup whose left cosets are equal to its right cosets is called a *normal* subgroup.

We use the notation $H \triangleleft G$ to indicate that H is a normal subgroup of G. Note that an abelian group has only normal subgroups, and the center of a group is always normal. The number of distinct cosets a subgroup has is called the *index* of the subgroup and is denoted $[G : H]$. In the case of a finite group, we have that

$$\frac{|G|}{|H|} = [G : H].$$

If H is a normal subgroup, we can construct the *quotient group* (or *factor group*) $\frac{G}{H}$, where the elements of $\frac{G}{H}$ are representatives of the cosets of H, with the group operation defined as $aH \circ bH = (a \circ b)H$.

Some classes of groups can be defined in terms of whether or not they possess a particular series of subgroups. Before we define these groups, we must first define the series in question. If G is a group, we may define $Z_0(G) = \{e\}$ and $Z_1(G) = Z(G)$. An *upper central series* is a series of subgroups such that

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \ldots$$

and $\frac{Z_{i+1}(G)}{Z_i(G)} = Z(G/Z_i(G))$.

A group is called *nilpotent* if there exists n such that $Z_n(G) = G$.

A group G is said to be *solvable* if there exist a series of subgroups $G_1, G_2, \ldots, G_n$ such that

1. $G_1 = \{e\}$
2. $G_n = G$
3. $G_i \lhd G_{i+1}$ for $i < 1 < n$
4. $\frac{G_{i+1}}{G_i}$ is abelian for $i < 1 < n$

Example 1.1. Any abelian group is solvable, since the sequence may be obtained by setting $G_1 = \{e\}$ and $G_2 = G$.

Example 1.2. If $G = D_4$, $G_1 = \{e\}$, $G_2 = \{e, R_{180}\}$, $G_3 = \{e, R_{90}, R_{180}, R_{270}\}$,$G_4 = G$ is such a sequence. Hence D_4 is solvable.

Every nilpotent group is solvable.

1.4 A Brief Introduction to `GAP`

As in many other areas of mathematics, computers have had an impact on the study of abstract algebra. As the name suggests, the field of *computational group theory* strives to use computers to answer questions about groups.[**?**] Several programs have been developed to deal with abstract algebra, but `GAP`[**?**] is one of the most prominent[**?**]. In the next section we will note previous research related to this topic performed with `GAP`. Later, `GAP` will be used in the exploration of groups which have subgroups satisfying properties in which we will be interested.

Chapter 2

Background on Group Covers

2.1 Introduction to Group Covers

A collection of proper subgroups whose union is equal to the entire group is called a *group cover*[**?**, **?**]. Our primary interest in this work will be on covers of finite groups. In particular, we will introduce and explore a particular type of finite group cover.

A result known as *Neumann's Lemma* characterizes groups that have finite covers.

Theorem 2.1 (Neumann's Lemma)**.** *Let $G = \bigcup_{i=1}^{n} g_i H_i$, where $H_1, \ldots, H_n$ are distinct (not necessarily distinct) subgroups of G. If we remove any cosets $g_i H_i$ which correspond to subgroups of infinite index, the union of the remaining cosets is still equal to G.*

Corollary 2.2. *A group has a finite covering by subgroups if and only if it has a finite homomorphic image which is not cyclic.*

Since any noncyclic finite group has itself as a noncyclic finite homomorphic image, it follows that every noncyclic finite group has a finite cover.

The fact that no group cover can consist of only two subgroups was first proven by Scorza, but it has been independently re-discovered many times [**?**]. One standard proof of this fact is elegant and instructive, so we include it here.

Proposition 2.3 (Scorza)**.** *No group can be written as the union of two proper subgroups.*

Proof. Suppose that H_1 and H_2 are proper subgroups such that $H_1 \cup H_2 = G$. Since H_1 and H_2 are proper, there must exist $x_1 \in H_1 - H_2$ and $x_2 \in H_2 - H_1$. Their product x_1x_2 must either be in H_1 or H_2. Without loss of generality, suppose $x_1x_2 \in H_1$. Since $x_1 \in H_1$, we know also that $x_1^{-1} \in H_1$. This tells us that $x_1^{-1}(x_1x_2) = (x_1^{-1}x_1)x_2 = x_2 \in H_1$, which is a contradiction. Hence no group can be written as the union of two proper subgroups. $\square$

This naturally leads to the question of determining how many subgroups are necessary to cover a particular group. The research done in this very active area will be the focus of the next section.

2.2 Minimal Covers

Before we discuss group covers any further, let us reduce slightly the type of covers we will consider. Specifically, we would like to avoid the case when the group cover has a proper subcollection which is also a group cover. To this end, an *irredundant* cover of G is defined to be one in which every member contains at least one element not contained in any other member of the cover [**?**]. We will only be concerned with irredundant covers. Cohn [**?**] introduced the convention that a group which can be covered by n proper subgroups, but no fewer, is called an *n-sum group*. In this case, n is called the covering number of G, denoted $\sigma(G) = n$ [**?**], and a covering by n subgroups is called a *minimal cover*. Every minimal cover is irredundant, but not vice versa.

The minimal covering number σ has attracted much study. By Proposition **??**, there is no group G such that $\sigma(G) = 2$. If a group has a minimal cover by three subgroups, then it must have a factor group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, also known as the Klein four-group [**?**, **?**]. Again, this fact was proven first by Scorza and re-discovered several times [**?**]. Cohn[**?**] showed that much information about the covering number of G can be determined by examining the covering number of factor groups of G; in particular, the covering number of G is always less than or equal to that of its factor groups. If $\sigma(G) = n$ and G has no normal subgroup N such that $\sigma(G/N) = n$, then G is called a *primitive n-sum group* [**?**]. It follows that if G is an n-sum group, it has a primitive n-sum group as a homomorphic image. In the same paper where he introduced the concept [**?**], Cohn presented several important facts concerning primitive n-sum groups. He completely described the primitive n-sum groups for the cases of n equal to 4,5, and $p+1$ for a prime p. In particular, Cohn showed that

1. If G is nilpotent, G is a primitive $(p+1)$-sum group if and only if it is $C_p \times C_p$.
2. $\sigma(G) = p+1$ for any non-cyclic p-group and any non-cyclic nilpotent group.
3. $C_2 \times C_2$ is the only primitive 3-sum group.
4. The primitive 4-sum groups are $C_3 \times C_3$ and S_3.
5. The only primitive 5-sum group is A_4.
6. There exists a group G with $\sigma(G) = p^n + 1$ for every prime power p^n.
7. If G is a primitive n-sum group, the center of G is trivial.

Bhargava [**?**] has recently shown that for each value of n, there is a corresponding finite (possibly empty) set of primitive n-sum groups.

Cohn [**?**] also conjectured that there were no 7-sum groups. Tomkinson confirmed this conjecture [**?**], and further conjectured that there would be no 11-sum, 13-sum, or 15-sum groups. This conjecture was confirmed in the case of 11, but both a 13-sum group [**?**] and a 15-sum group [**?**] have been found. These inquiries led to the question of which other numbers occur as the covering number of a group. For solvable groups, an important class of groups related to the historical development of abstract algebra [**?**], Tomkinson showed the covering number is always $p^n + 1$ for some prime p and some integer n [**?**]. Recently, Garonzi [**?**] completely characterized all groups which have $\sigma \leq 25$ and shows that there are no n-sum groups for $n \in \{19, 21, 22, 25\}$. The question of which integers occur as group covering numbers remains open for most values larger than 25 and not of the form $p^n + 1$.

Where precise values for $\sigma(G)$ have yet to be obtained, upper and lower bounds may be of interest. The number of subgroups involved in *any* cover of G provides an upper bound on $\sigma(G)$. Maroti [**?**], as well as Kappe and Redden [**?**], have provided values and bounds for the covering numbers of the symmetric and alternating groups up to degree

10. Kappe and Redden have used `GAP` extensively in establishing their results [?]. Based on our personal experience, we believe that if $n \geq 11$, A_n is too large to explore with `GAP` on a personal computer at this time. We should note that Maroti[?] has established the precise value of $\sigma(S_n)$ in the case that n is an odd number not equal to 9, an upper bound on the value of $\sigma(S_n)$ in the case that n is even, and an upper bound for $\sigma(A_n)$ in the case that n is not equal to 7 or 9.

Many authors have investigated the σ-values for certain classes of groups, and this topic has been particularly active in recent years. Serena has collected and summarized many results from the 20th century[?]. For a thorough description of recent results, the reader is referred to the introduction in [?].

2.3 Group Covers by Subgroups with Given Properties

A related focus of research activity is the existence of coverings of groups by subgroups which share a certain property. G.A. Miller is believed to have instigated the study of group covers, with a 1901 paper [?] which explored groups whose covers have trivial pairwise intersection. Such group covers eventually came to be known as *partitions* [?]. Miller showed that the only such abelian groups were elementary abelian groups; he also showed that the members of such a partition must have equal order.

While the partitions of abelian groups were settled by Miller's brief note, for non-abelian groups the matter is more complicated and is still not completely settled [?]. Several significant group theorists, including Hughes, Baer, and Suzuki, have been linked with the study of partitions [?, ?].

Bryce and Serena [?] determined groups which have minimal covers consisting of abelian subgroups. Bhargava [?] has shown that a group can be covered by proper normal subgroups if and only if it has a quotient group isomorphic to $C_p \times C_p$ for some p; he calls such groups *anti-simple*. Brodie and Kappe [?] investigated groups which have a covering by subgroups with group properties closely related to commutativity. Foguel and Ragland [?] investigated which classes of groups can be covered by *isomorphic* abelian subgroups. They used `GAP` in parts of this work. The 2002 survey article by Serena [?] covers many of the group covers by subgroups with certain properties.

Chapter 3

Introduction to Equal L-Intersection Group Covers

Our objective is to explore a new class of group covers. In particular, we require that the members of the cover be subgroups of equal order and have equal but nontrivial pairwise intersection. We will be especially interested in the case when this pairwise intersection is isomorphic to a cyclic group of prime order.

The intersection of subgroups arising in certain types of covers has been touched upon in some previous work. In particular, Greco [**?**] considered groups which could be covered by five subgroups with equal pairwise intersection. Neumann [**?**] considered a function $f(n)$ defined to be the maximal value of $[G : \cap_{i=1}^{n} H_i]$, where $\{H_i\}_{i=1}^{n}$ is an irredundant cover of G. Tomkinson [**?**] established an upper bound for this function. Bryce, Fedri, and Serena [**?**] showed that $f(5) = 16$.

In these works, the subgroup intersections (whether pairwise or total) were considered as a side effect of determining group covers with certain properties. In our work, the pairwise intersection is given primary importance, and we investigate the number and type of subgroups involved the covers which occur as a result.

We begin by relating this class of group covers to other types of group covers. While the class we will define overlaps with these covers, we will show that it is not identical to any of them.

3.1 Known Classes of Equal Covers

In this section we will examine some existing work on group covers whose members are of equal order.

Definition 3.1. Let G be a group. An *equal partition* of G is a collection of subgroups $\{H_1, H_2, \ldots, H_n\}$ such that

1. $G = \bigcup_{i=1}^{n} H_i$ (the collection is a group cover)
2. $|H_i| = |H_j|$ for $1 \leq i, j, \leq n$ (all subgroups are the same size)
3. $H_i \cap H_j = \{e\}$ for $1 \leq i, j \leq n$ and $i \neq j$ (the pairwise intersection is trivial).

Definition 3.2. Let G be a finite group with an equal partition $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. If $H_i H_j = G$ for $1 \leq i, j, \leq n, i \neq j$, $\mathcal{H}$ is called a *spread* of G.

The following result, due to Miller [**?**], provided one of the first results on group covers of any type.

Theorem 3.3. *Let G be an abelian group. There exists an equal partition of G if and only if G is an elementary abelian group.*

Proof. Let G be a finite group. We need to show that G has order p^k for some prime p and some integer k, and that the exponent of G is equal to p.

Let $\{H_i\}_{i=1}^{n}$ be an equal partition of G. Let x, y be non-identity elements of G such that $x \in H_j, y \in H_k$ and $j \neq k$. First, we will show that $xy \in H_r$ for some $1 \leq r \leq n$ such that $r \neq j, k$. If $xy \in H_j$, then $(x)^{-1}xy = y \in H_j$, hence $y \in H_j \cap H_i$, and the pairwise intersection is nontrivial. A similar contradiction arises if $xy \in H_k$.

Next, we will show that $|G|$ is a prime power. To obtain a contradiction, suppose that $|G|$ is divisible by distinct primes p_1 and p_2. By Cauchy's Theorem, we can select $a \in G$ such that $|a| = p_1$ and $b \in G$ such that $|b| = p_2$ such that a and b are contained in distinct subgroups H_i and H_j, and, without loss of generality, assume $p_1 > p_2$. As noted above, ab must be in a subgroup H_r distinct from H_i and H_j. Then we have $(ab)^{p_2} = a^{p_2} \neq e$. As a power of ab, $a^{p_2} \in H_r$; however, since it is a power of a, $a^{p_2} \in H_a$. Since H_a and H_{ab} must be distinct, this contradicts our assumption that the intersection of distinct subgroups contains only the identity. Thus, the order of an element in G can only be divisible by a single prime p, which means that for all $g \in G$, $|g| = p^k$

Now we will show that $k < 2$. To this end, we will show that G does not contain an element of order p^r for $2 \leq r$. Again, for contradiction's sake, suppose there exist $a \in G$ such that a has order p and $b \in G$ such that b has order p^r. Again, let $a \in H_i$ and $b \in H_j$. Then $ab \in H_r$, with $(ab)^p = a^p b^p = b^p \in H_r \cap H_j$, which contradicts our assumption that distinct subgroups have trivial pairwise intersection. This means that all non-identity elements in G have order p; hence G is elementary abelian. □

Isaacs [**?**] provided the following results which characterized *all* groups with an equal partition.

Theorem 3.4. *Let G be a finite group. G has an equal partition if and only if G is a p-group of exponent p. Also, G has a spread if and only if it is elementary abelian.*

Isaacs' paper concludes with the following question, which we have reason to believe is still open[**?**]:

Question 3.5. *Does there exist a group with an equal partition in which not all of the subgroups are abelian?*

Foguel and Ragland [**?**] investigated a class of groups covered by isomorphic abelian subgroups; such groups are known as *CIA*-groups. Obviously, isomorphic subgroups must have equal order. However, the definition of *CIA*-groups places no restriction on the pairwise intersection of the subgroups. While the pairwise intersections must be isomorphic subgroups, they may not be equal in the set-theoretic sense.

We would note that every equally partitionable group is a *CIA*-group, since it is covered by cyclic subgroups of order p, and cyclic subgroups of prime order are abelian groups unique up to isomorphism [**?**].

The following theorem tells us how to tell when an abelian group is a *CIA*-group.

Theorem 3.6. *Abelian groups are* CIA*-groups if and only if they have $C_{p^n} \times C_{p^n}$ as a direct factor for some prime p and some integer n.*

Foguel and Ragland also asked the following:

Question 3.7. *Does there exist an* CIA-*group which has a trivial center?*

Recall that Cohn showed that non-abelian primitive n-sum groups always have a trivial center. Thus, a negative answer to this question would imply that the classes of non-abelian primitive n-sum group and *CIA*-groups are mutually exclusive.

3.2 Definition and Examples of Equal L-Intersection Cover

It seems natural to generalize the notion of an equal partition to covers whose members are equal but nontrivial. This leads to the following definition:

Definition 3.8. Let G and L be finite groups. If there exist subgroups $H_1, H_2, \ldots, H_n$ and a subgroup K of G such that

- $G = \bigcup_{i=1}^{n} H_i$ (the subgroups are a group cover)
- $|H_i| = |H_j|$ for $1 \leq i, j \leq n$ (the subgroups are of equal size)
- $H_i \cap H_j = K$ for $1 \leq i, j, \leq n, i \neq j$
- $K \cong L$

we say that G has an equal L-intersection cover.

We have a similar generalization of the notion of a spread.

Definition 3.9. An *L-intersection spread* is an L-intersection cover such that $H_i H_j = G$ for $1 \leq i, j, \leq n, i \neq j$.

In our terminology, an equal partition can be called an equal $\{e\}$-intersection cover, while a spread would be an $\{e\}$-intersection spread.

Example 3.10. The Dihedral group of order 8 has an equal C_2-intersection cover consisting of $\{R_0, R_{90}, R_{180}, R_{270}\}, \{R_0, R_{180}, V, H\}$, and $\{R_0, R_{180}, D, D'\}$.

This was our original motivating example, as it shows that a group which has an equal L-intersection cover need not be a *CIA*-group. However, certain results about *CIA*-covers have nice analogs for equal L-intersection covers. For instance, Foguel and Ragland show how to construct a *CIA*-covering for the direct product of two *CIA*-groups. Mimicking their approach, we can show how to construct a group which has an equal L-intersection cover when we know that L is isomorphic to the direct product of two groups L_1 and L_2.

Theorem 3.11. *Let G_1 and G_2 be finite groups. If G_1 has an equal L_1-intersection cover and $G_2 \cong L_2$, then $G_1 \times G_2$ has an equal $L_1 \times L_2$-intersection cover.*

Proof. Suppose that G_1 is a finite group with a cover by subgroups $\{H_i\}_{i=1}^n$ which have equal order and pairwise intersection equal to K, where K is isomorphic to L_1. Then taking the direct product of G_1 and G_2, we have the following.

$$\begin{aligned} G_1 \times G_2 &= \left(\bigcup_{i=1}^n H_i\right) \times G_2 \\ &= \bigcup_{i=1}^n (H_i \times G_2) \end{aligned}$$

Hence $G_1 \times G_2$ is covered by the n subgroups $H_i \times G_2$. Since $|H_i| = |H_j|$, we have $|H_i \times G_2| = |H_i||L_2| = |H_j||L_2| = |H_j \times G_2|$; hence these subgroups are of equal order. Also,

$$\begin{aligned} (H_i \times L_2) \cap (H_j \times L_2) &= (H_i \cap H_j) \times (L_2 \cap L_2) \\ &= (H_i \cap H_j) \times L_2 \\ &= L_1 \times G_2 \\ &\cong L_1 \times L_2 \end{aligned}$$

This completes the proof. □

Since every abelian group can be expressed as the direct product of cyclic groups, this gives us a way to construct a group which has an L-intersection cover for any abelian L.

First, we need to be able to find a C_n-intersection for any choice of n. The following result allows us to do that.

Corollary 3.12. *If G_1 is a p-group of exponent p, then $G_1 \times G_2$ has an equal G_2-intersection cover.*

Proof. We know from Theorem **??** that since G_1 is a p-group of exponent p, G_1 has an equal partition. By definition, an equal partition is an equal $\{e\}$-intersection cover. Hence $G \times G_2$ has an $\{e_{G_1}\} \times G_2$-intersection cover. Since $\{e_{G_1}\} \times G_2$ is isomorphic to G_2, this gives a G_2-intersection cover. $\square$

Example 3.13. Applying the corollary, we see that $C_2 \times C_2 \times D_4$ has an equal D_4-intersection cover by 3 groups of order 16 isomorphic to $C_2 \times D_4$. This is an example of a group which is covered by isomorphic *non*-abelian subgroups.

Corollary 3.14. *Let G be an elementary abelian group of order p^n. G has an equal $(C_p)^i$-intersection cover for $1 \leq i \leq (n-2)$.*

Proposition 3.15. *Let n be a square-free integer, i.e. $n = p_1 p_2 \ldots p_k$, where $p_1, \ldots, p_k$ are distinct prime numbers. Let $m_i = p_1 \ldots p_{i-1} p_{i+1} \ldots p_k$ for $1 \leq i \leq k$. If $G = C_n \times C_n$, G has an equal $C_{m_i} \times C_{m_i}$-cover for each m_i.*

Proof. We know that $C_n \times C_n \cong C_{p_1 p_2 \ldots p_n} \times C_{p_1 p_2 \ldots p_n}$. Further, since the p_i are all relatively prime, $C_{p_1 p_2 \ldots p_n}$ is isomorphic to $C_{p_i} \times C_{p_1} \times C_{p_2} \times \ldots C_{p_{i-1}} \times C_{p_{i+1}} \cdots \times C_{p_n}$. Hence we have

$$\begin{aligned} C_{p_1 p_2 \ldots p_n} \times C_{p_1 p_2 \ldots p_n} &\cong (C_{p_i} \times C_{p_i}) \\ &\times \big(C_{p_1} \ldots C_{p_{i-1}} \times C_{p_{i+1}} \cdots \times C_n\big) \\ &\times \big(C_{p_1} \ldots C_{p_{i-1}} \times C_{p_{i+1}} \cdots \times C_n\big) \\ &\cong (C_{p_i} \times C_{p_i}) \times (C_{m_i} \times C_{m_i}) \end{aligned}$$

We know that $C_{p_i} \times C_{p_i}$ is a p-group of exponent p. Thus it has an equal partition, i.e. an $\{e\}$-intersection cover, and we can apply **??** to achieve the desired result. $\square$

Proposition 3.16. *Let p be a prime number. If $G = C_{p^2} \times C_p$, then G has an equal C_p-intersection cover.*

Proof. For $0 \leq i \leq (p-1)$, let $H_i = \langle (1,i) \rangle$. Let $H_p = \langle (p,0),(1,0) \rangle$. Each subgroup has order p^2, and their pairwise intersections correspond to the subgroup $K = \langle (p,0) \rangle$, which is a cyclic group of order p. $\square$

For various choices of L, we have shown some methods and examples of how to find an equal L-intersection cover. These covers may allow us to explore open questions related to equal partitions.

Remark 3.17. Let G be a finite group with an equal L-intersection cover $\{H_i\}_{i=1}^n$, where $H_i \cap H_j = K$ and $K \cong L$. If K is a normal subgroup of G and there exists $1 \leq k \leq n$ such that both H_k and H_k/K are non-abelian, an example to satisfy Question **??** will have been found.

3.3 Methods used to find equal L-intersection covers

The previous section established how to construct equal L-intersection covers for various choices of L. Now we turn our attention to the following problem: given finite groups G and L, determine whether or not G has an equal L-intersection cover. Throughout the remainder of this work, all groups are finite unless otherwise specified.

Let G and L be finite groups. We would like to be able to determine whether or not G has an equal L-intersection cover. Thus, at the beginning of our investigation, G and L (and hence $|G|$ and $|L|$) are known. A first step is to determine properties which the number and order of subgroups in such a cover must possess. This will give us tests to determine whether or not such a cover or spread is even possible. We can use these tests to help automate the search for groups with equal L-intersection covers, as well as to demonstrate that certain groups do not have such a cover.

3.3.1 Properties of the Size and Number of Subgroups

Our objective in this section is to determine what can be said about the number and size of the subgroups involved in an equal L-intersection cover. Throughout, n will denote the number of subgroups in the cover, while $|H|$ will denote the size of any such subgroup.

First we present a proof that a finite group can not be covered by two proper subgroups. This proof relies only on some counting arguments and Lagrange's Theorem. We have not encountered it in the existing literature. While we have already demonstrated a stronger result which holds for all groups, we include this proof because it illustrates the types of counting and divisibility arguments which we will utilize throughout this section.

Theorem 3.18. *No finite group can be written as the union of two proper subgroups.*

Proof. Let G be a finite group. Suppose there exist proper subgroups A and B such that $G = A \cup B$. Let $M = A \cap B$. Let $a = |A|, b = |B|$, and $m = |M|$. Note that $a, b > m$, otherwise $M = A = B = G$. Also note that since M is the intersection of two subgroups, it must also be a subgroup; hence $m > 0$.

By the Inclusion-Exclusion Principle, we know that

$$|G| = |A| + |B| - |A \cap B| = a + b - m.$$

First, we will show that $a \neq b$. To obtain a contradiction, assume $a = b$. By Lagrange's Theorem, there exist positive integers c_1, c_2 such that $c_1 a = |G| = a + b - m$ and $c_2 b = |G| = a+b-m$. Since $a = b$, we have $c_1 a = a+a-m = 2a-m$, hence $m = a(2-c_1)$. Since A is a proper subgroup, we must have $c_1 > 1$. If $c_1 = 2$, $m = 0$, which is a contradiction. If $c_1 > 2$, m is negative, which is also a contradiction. Thus, in the above equation $a \neq b$.

Now, assume without loss of generality that $b > a$. Since $|G| = a+b-m$, by Lagrange's Theorem we know that a divides $a + b - m$, hence a divides $b - m$. Similarly, b divides $a - m$. Hence there exist n_1, n_2 such that $n_1 a = b - m$ and $n_2 b = a - m$, which implies that $b = n_1 a + m$ and $n_2 b + m = a$. Substituting the first equation into the second, we obtain that

$$n_2(n_1 a + m) + m = a$$

. Then

$$n_2 n_1 a + n_2 m + m = a,$$

where $n_1, n_2, m > 0$. However, this gives us the following chain of relationships

$$\begin{aligned}
|A| &= n_2 n_1 a + n_2 m + m \\
&> n_2 n_1 a + n_2 m \\
&\geq n_2 n_1 a + n_2 \\
&> n_2 n_1 a \\
&\geq n_1 a \\
&\geq a \\
&= |A|
\end{aligned}$$

This implies that $|A| > |A|$, a contradiction. This completes the proof. $\square$

The following result, due to Cohn, gives the first condition relating the number of subgroups n and the size of subgroups $|H_i|$ in a group cover. In Cohn's notation, the subgroups are arranged such that $|H_i| \leq |H_j|$ for $i < j$. Thus we assume that H_1 has order equal to the minimum order of all the subgroups in the cover.

Proposition 3.19 (Cohn)**.** *If $G = \bigcup_{i=1}^{n} H_i$, then $|G| \leq \sum_{i=2}^{n} |H_i|$, with equality if and only if*

a) $H_1 H_r = G$ for $1 \neq r$, and

b) $H_r \cap H_s \subset H_1, r \neq s$ for all $1 \leq r, s \leq n$

Proof. Suppose $H_r \neq H_1$. We need to count the number of elements contained in H_r which are not contained in H_1. Obviously, this number is equal to $|H_r| - |H_1 \cap H_r|$.

We know that

$$\frac{|H_1||H_r|}{|H_1 \cap H_r|} = |H_1 H_r|,$$

which we can manipulate to obtain

$$|H_1 \cap H_r| = \frac{|H_1||H_r|}{|H_1 H_r|}.$$

Hence,

$$\begin{aligned} |H_r| - |H_1 \cap H_r| \quad &= |H_r| - \tfrac{|H_1||H_r|}{|H_1H_r|} \\ &= |H_r| \left(1 - \tfrac{|H_1|}{|H_1H_r|}\right) \end{aligned}$$

We know that $|H_1H_r| \leq |G|$, which clearly becomes an equality if and only if $|H_1H_r| = G$. Then $\frac{|H_1|}{|G|} \leq \frac{|H_1|}{|H_1H_r|}$, so $1 - \frac{|H_1|}{|G|} \geq 1 - \frac{|H_1|}{|H_1H_r|}$, hence

$$|H_r| - |H_1 \cap H_r| \leq |H_r|(1 - \frac{|H_1|}{|G|}).$$

Now we know that

$$|G| \leq |H_1| + \sum_{r=2}^{n} |H_r - H_1| \leq |H_1| + (1 - \frac{H_1}{G} \sum_{r=2}^{n} |H_r|).$$

The first inequality in the line above is an equality and if only if the intersection of any two distinct subgroups H_i, H_j for $2 \leq i, j \leq n$ is contained in H_1.

Then we have

$$\begin{aligned} \frac{|G| - |H_1|}{1 - \frac{|H_1|}{|G|}} &\leq \sum_{r=2}^{n} |H_r| \\ \frac{|G| - |H_1|}{\frac{|G|-|H_1|}{|G|}} &\leq \sum_{r=2}^{n} |H_r| \\ |G| &\leq \sum_{r=2}^{n} |H_r| \end{aligned}$$

□

Note that in the case of an equal L-intersection cover, all subgroups have equal index, and hence the arrangement of the subgroups in the cover is arbitrary. This allows any group to play the role of H_1. We note also that by definition of an equal L-intersection cover, condition (b) is always satisfied. This means that for the type of cover we are investigating, we can say the following:

Corollary 3.20. *If G has an equal L-intersection cover by n subgroups of the same order as order H, then $|G| \leq (n-1)|H|$, with equality if and only if the H_i form a L-intersection spread.*

Corollary 3.21. *If G is a group with an equal L-intersection cover by n subgroups of order $|H|$, then $n \geq 1 + \dfrac{|G|}{|H|}$, with equality if and only if the cover is an L-intersection spread.*

The above results hold for all covers by subgroups of equal order, regardless of their pairwise intersections. Now we will explore the restrictions related to n and $|H|$ which result from our choice of L.

Theorem 3.22. *If G is a group with an equal L-intersection cover by n subgroups of order $|H|$, then $|G| = n|H| - (n-1)|L|$.*

Proof. Since the $\{H_i\}_{i=1}^n$ are a cover of G we have $|G| = |\bigcup_{i=1}^n H_i|$. Applying the Inclusion-Exclusion Principle and noting that all pairwise intersections, and hence all intersections for any choice of $k > 2$ subgroups, are equal, we have

$$\begin{aligned}
\left|\bigcup_{i=1}^n H_i\right| &= \sum_{i=1}^n |H_i| - \binom{n}{2}|H_i \cap H_j| + \binom{n}{3}|H_i \cap H_j \cap H_l| - \ldots (-1)^n \bigcap_{i=1}^n H_i \\
&= \sum_{i=1}^n |H_i| - \binom{n}{2}|L| + \binom{n}{3}|L| + \cdots + (-1)^n \binom{n}{n}|L| \\
&= (\sum_{i=1}^n |H_i|) - |L| \left(\sum_{i=2}^n (-1)^i \binom{n}{i}\right)
\end{aligned}$$

Recalling that $|H_1| = |H_i|$ and letting $x = \sum_{i=2}^n (-1)^i \binom{n}{i}$, we we can rewrite this as

$$|G| = (n-1)|H_i| - |L|x.$$

Our goal is to express this x in terms of n. Note that by the Binomial Theorem, we can say

$$\begin{aligned}
0 &= (1-1)^n \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \\
&= 1 - n + \sum_{i=2}^{n} (-1)^i \binom{n}{i} \\
(n-1) &= \sum_{i=2}^{n} (-1)^i \binom{n}{i}
\end{aligned}$$

Hence $x = (n-1)$, and the desired result follows. $\square$

We can combine this result with our corollary to Theorem **??** to obtain more information about the relationship between n and $|H|$.

Corollary 3.23. *If $\{H_i\}_{i=1}^n$ are an L-intersection cover for G, $n \geq 1 + \frac{|H|}{|L|}$, with equality if and only if the H_i are an L-intersection spread.*

Proof. Let G be a group with an equal L-intersection cover by n subgroups of order $|H|$. We know from Theorem **??** that $G = n|H| - (n-1)|L|$, while we know from Proposition **??** that $G \leq (n-1)|H|$. Combining these two results, we obtain

$$\begin{aligned}
n|H| - (n-1)|L| &= |G| \\
n|H| - (n-1)|L| &\leq (n-1)|H| \\
n|H| - (n-1)|L| &\leq n|H| - |H| \\
-(n-1)|L| &\leq -|H| \\
(n-1)|L| &\geq |H|
\end{aligned}$$

The inequality in the second line is an equality if and only if the H_i form an L-intersection spread. $\square$

Corollary 3.24. *If G is a group with an equal L-intersection cover by n subgroups, $n \geq 1 + \sqrt{\frac{|G|}{|L|}}$. If the cover is an L-intersection spread, $n = 1 + \sqrt{\frac{|G|}{|L|}}$.*

Proof. Let G be a group with an equal L-intersection cover by subgroups of order $|H|$. Combining the above results, we know that $|G| \leq (n-1)|H| \leq (n-1)(n-1)|L| = (n-1)^2|L|$. Since $|G| \leq (n-1)^2|L|$, we can obtain divide both sides by $|L|$, take the square root of both sides, and add 1 to achieve the desired result. $\square$

Proposition 3.25. *If G is a group with an equal L-interseection cover by subgroups of order $|H|$, then $\sqrt{|G||L|} \geq |H|$, with equality if and only if the cover is an L-intersection spread.*

Proof. Let G be a subgroup. Let H_i and H_j be arbitrary elements of an equal L-intersection cover. Then

$$|G| \geq |H_iH_j| = \frac{|H_i||H_j|}{H_i \cap H_j} = \frac{|H|^2}{|L|},$$

hence

$$|G||L| \geq |H|^2,$$

and

$$\sqrt{|G||L|} \geq |H|.$$

Again, $|G| = |H_iH_j|$ if and only if the H_i form an L-intersection spread. $\square$

This means that if G has an L-intersection spread, the size of the subgroups in the spread are completely determined by the choice of L. Since for an L-intersection spread, $n = \sqrt{\frac{|G|}{|L|}} + 1$, this means the value of n is determined by the choice of L, as well.

Corollary 3.26. *If G has an equal L-intersection cover,* $\max\{|g||g \in G\} \leq \sqrt{|L||G|}$.

Proof. The order of each element must divide the H_i, and hence $|g| \leq |H| \leq \sqrt{|L||G|}$. $\square$

For a fixed subgroup size $|H|$, these bounds give us information about n. Likewise, for a fixed cover size n, we have information about potential subgroup sizes. It would be nice to have information about the possible values of n which depends only on $|G|$ and $|L|$. This next result allows us to do exactly that.

Proposition 3.27. *If G has an equal L-intersection cover by n subgroups of order equal to $|H|$, n is a proper divisor of $|G| - |L|$.*

Proof.

$$|G| = n|H| - (n-1)|L|$$
$$|G| = n|H| - n|L| + |L|$$
$$|G| - |L| = n(|H| - |L|)$$

This shows that n divides $|G| - |L|$. Since $|L|$ divides $|H|$, $|H| - |L|$ must be an integer greater than 1, and hence n is a proper divisor of $|G| - |L|$. □

Proposition 3.28. *If G has an equal L-intersection cover by n subgroups of order $|H|$, $|H|$ divides $(n-1)|L|$.*

Proof. Again, we begin with the fact that $|G| = n|H| - (n-1)|L|$. Since $|H|$ is a divisor of $|G|$ by Lagrange's Theorem, reducing both sides $\mod |H|$, we have

$$0 \equiv -(n-1)L(\mod |H|)$$
$$0 \equiv -nL + L(\mod |H|)$$
$$nL \equiv L(\mod |H|)$$

□

Combining the previous propositions, we can state the following theorems:

Theorem 3.29. *Let G be a finite group. If $H_1, H_2, \ldots, H_n$ is an equal L-intersection cover for G, then*

- $n \geq \max\{\frac{|G|}{|H|} + 1, \frac{|H|}{|L|} + 1, \sqrt{\frac{|G|}{|L|}} + 1\}$
- $\sqrt{|G||L|} \geq |H| \geq \max\{|g| : g \in G\}$
- $|G| \equiv |L|(\mod n)$
- $n|L| \equiv |L|(\mod |H|)$

Theorem 3.30. *Let G be a finite group. If G has subgroups $H_1, H_2, \ldots, H_n$ which form an equal L-intersection spread for G, then*

- $n = \frac{|G|}{|H|} + 1 = \frac{|H|}{|L|} + 1 = \sqrt{\frac{|G|}{|L|}} + 1$
- $\sqrt{|G||L|} = |H| \geq \max |g||g \in G$
- $|G| \equiv |L| (\mod n)$
- $n|L| \equiv |L| (\mod |H|)$

Now we will consider the special case when G is a p-group. Since every subgroup of G is a p-group, we may assume that $|G| = p^a$, $|H| = p^b$, and $|L| = p^c$ for integers a, b, and c such that $a > b > c$. We can apply the results in the above theorems to illuminate some specific cases.

1. We can apply the fact that $\sqrt{|G||L|} \geq |H|$. In this instance, we have that

$$\begin{aligned} \sqrt{p^a p^c} &\geq p^b \\ (p^a p^c)^{\frac{1}{2}} &\geq p^b \\ (p^{a+c})^{\frac{1}{2}} &\geq p^b \\ p^{\frac{a+c}{2}} &\geq p^b \\ \frac{a+c}{2} &\geq b \end{aligned}$$

   As before, this inequality is an equality if and only if $H_i H_j = G$ for $1 \leq i, j \leq n, i \neq j$. This tells us that if $a + c$ is odd, G does not have an L-intersection spread.

2. From the above, we can say that the maximum order of the elements in G must be less than or equal to $p^{\frac{a+c}{2}}$.

3. If the subgroups H_i in the cover are assumed to be maximal, then the H_i must be an L-intersection spread. Also, we know that the H_i must have order p^{a-1}. Then we have $\frac{a+c}{2} = a-1$, hence $a+c = 2(a-1) = 2a-2$, which implies that $c = a-2 = b-1$.

4. If L is assumed to be minimal (i.e. isomorphic to C_p), then we know that $c = 1$. Hence $\frac{a+1}{2} \geq b$, or $a \geq 2b - 1$, with equality if the H_i are an L-intersection spread.

5. If the H_i are maximal and L is minimal, $c = b - 1$ becomes $1 = (a - 1) - 1$. This means $a = 3$.

In the previous section, we determined examples of abelian groups which have C_p-intersection covers for some prime p. Now we will use the above results to show that some abelian groups do not have such a cover.

Proposition 3.31. *$C_{p^k} \times C_p$ does not have an equal C_p-intersection cover for $k > 2$.*

Proof. In this case, $|G| = p^{k+1}$ and $|L| = p$. However, G has elements of order p^k, and

$$p^k = p^{\frac{2k}{2}} = p^{\frac{k+k}{2}} > p^{\frac{k+2}{2}} = \sqrt{|G||L|}$$

. □

3.3.2 Functions in `GAP` to find equal L-intersection covers

Applying the results in the previous section, we can implement algorithms in GAP to compute whether or not a certain group G has an equal L-intersection cover. The first algorithm in `GAP` is called `EqualLIntersectionCoverValues`; it takes a group G and an isomorphism class for the subgroup L and returns a list of ordered pairs $(n, |H|)$ which could be the n and $|H|$ values for an equal L-intersection cover.

The `SmallGroups` library in `GAP` contains data about all small groups of order less than 2000, except for groups of order 1024. Using this library, we can apply our commands to groups in a given small range.

First, we need a way for `GAP` to find subgroups of a given group. Because large groups have many subgroups, `GAP` focuses first on finding the conjugacy classes of subgroups.[**?**] As the name suggests, the command

```
ConjugacyClassesSubgroups(G)
```

returns the conjugacy classes of subgroups of G. Since every subgroup of G is contained in some conjugacy class, the command

```
Union(ConjugacyClassesSubgroups(G))
```

returns all the elements in every conjugacy class, i.e. all subgroups of G.

Then we can use the following command in GAP to find all subgroups of a given group of a particular size k.

```
kSubgroups := function(G,k);
return Filtered(Union(
       ConjugacyClassesSubgroups((G)),
       x -> Size(x) = k);
```

The following command iterates over these potential values and sizes to determine whether or not G does in fact possess an equal L-intersection cover. Its logic is based on the results of the previous section–by testing to see whether or not certain values obey the conditions we have established, we can determine whether or not it is possible for a group to have an equal L-intersection cover.

5in

```
EqualLIntersectionCoverValues := function(G,L);
local g,l,maxorder,n1,n2,N,H,i,subgrps,values;


g := Size(G);
l := Size(L);
maxorder := Maximum(List(G, x -> Order(x)));
values := List([]);


if  maxorder > RootInt(g * l, 2) then
   return values;
fi;


n1 := [RootInt(g)/RootInt(l) .. (g - l) - 1];
n2 := DivisorsInt(g - l);
```

```
N := Intersection(n1,n2);
H := List(N, x -> (g + (x - 1)*l) / x);
subgrps := Union(ConjugacyClassesSubgroups(G));

for i in [1 .. Size(H)] do
   if (H[i] in DivisorsInt(g)) and
        (l in DivisorsInt(H[i])) then
     if (maxorder <= H[i])
         and (H[i] in DivisorsInt((N[i]-1) * l)) then
       if kSubgroups(G,H[i]) >= N[i] then
        if RootInt(g*l) >= H[i] then
         Add(values,[N[i],H[i]]);
        fi;
      fi;
    fi;
  fi;
od;

return values;

end;
```

Using that command, we implement a `GAP` function `EqualLIntersectionCovers` which, given a group G and an isomorphism class L, finds L-intersection covers of G. For each potential $(n, |H|)$ pair returned by `EqualLIntersectionCoverValues`, the algorithm considers every potential combination of n subgroups of order H, removing first those which are not group covers, then those whose total intersection is not isomorphic to L, then finally those which do not have equal pairwise intersections. The remaining collections are the desired covers. The code for the algorithm is as follows.

5in

```
EqualLIntersectionCovers := function(G,L)
local values,covers,subgrps,i,j,colls,pairs,intersects;

Read("EqualLIntersectionCoverValues.txt");
covers := [];
values := EqualLIntersectionCoverValues(G,L);
for i in values do
   subgrps := kSubgroups(G,i[2]);
   colls := Combinations(subgrps,i[1]);
   colls := Filtered(colls, x -> IsSubset(Union(x),G));
   for j in colls do
      if StructureDescription(Intersection(j))
            = StructureDescription(L) then
        pairs := Combinations(j,2);
        intersects = List(pairs, x -> Intersection(x));
        if Size(Collected(intersects)) = 1
            and Flat(Collected(intersects) = Size(L)
            then
          Add(covers,j);
        fi;
      fi;
    od;
od;
return covers;
end;
```

The main advantage of this algorithm is that for a choice of L, it explicitly determines all such covers and can be automated to search all groups of a given size. The disadvantage is that it is memory-intensive. If we suppose that G has r subgroups of order $|H|$. Unfortunately, generating all $\binom{r}{n}$ collections of n subgroups of order $|H|$ can be memory-intensive for large values of r. We will quickly encounter circumstances where this restriction renders this algorithm impractical. In certain circumstances, there are ways to work around these difficulties. However, the computational approach will allow us to find many examples.

Chapter 4

Finding 2-groups with equal C_2-intersection covers

The previous section developed computational methods to search for equal L-intersection covers. In this section, we will give the results of applying those methods to a particular class of groups.

4.1 Results on Equal C_2-intersection covers for some 2-groups

We will be exploring groups of order 2^k for some positive integer k to see which ones have an equal C_2-intersection cover. In this case, since $|L| = 2$, one previously proven result becomes very sharp. We have shown already that if G has an equal L-intersection cover by n subgroups of order $|H|$, $|H|$ must be a divisor of $n|L| - |L|$. If $|L| = 2$, it follows that $|H|$ must be a divisor of $2n - 2 = 2(n - 1)$. Hence $\frac{|H|}{2}$ must be a divisor of $(n - 1)$. Since H is a power of 2, this tells us that n must be odd.

After examining the data about these groups acquired computationally, we will present conjectures and questions inspired by these examples.

4.1.1 Groups Of Order 8

There are 4 noncyclic groups of order 8: the dihedral group of order 8 s(D_4), the quaternions (Q_8), $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$. Each one has an equal C_2-intersection cover by subgroups of order 4. We know that $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$ have equal C_2-intersection covers by theorems in Section 1. We gave an equal C_2-intersection cover for D_4 as the first example of such a cover. The quaternions are covered by the equal-order subgroups $H_1 = \{1, -1, i, -i\}, H_2 = \{1, -1, j, -j\}$, and $H_3 = \{1, -1, k, -k\}$, which have pairwise intersection $\{1, -1\} \cong C_2$. This is the only instance which we were able to determine completely by hand.

4.1.2 Groups of order 16

Using the `GAP` algorithm `EqualLIntersectionCovers` which we previously introduced, we search for all groups which have equal C_2-intersection covers.

Since the number of subgroups n in such a cover must divide $16 - 2 = 14$ and $n \neq 2$, n must be equal to 7. Hence 7, which can not occur as a minimal covering number, does occur as an equal L-intersection covering number.

We can eliminate the cyclic group C_{16} from consideration. Similarly, we know that $C_2 \times C_2 \times C_2 \times C_2$ will have an equal C_2-intersection cover, and we do not force `GAP` to look for it.

Our `GAP` algorithm tells us that the following groups of order 16 have equal C_2-intersection covers.

1. $C_4 \times C_2 \times C_2$
2. $C_2 \times D_4$
3. $C_2 \times Q_8$
4. $(C_4 \times C_2) \ltimes C_2$

We note that every group in this list occurrs as either the direct product or the semi-direct product of C_2 and a group of order 8 with an equal C_2-intersection cover. In each case, `GAP` found only one equal C_2-intersection cover for each group.

4.1.3 Groups of order 32

Again, we begin by determining the value of n, the number of subgroups involved in an equal L-intersection cover. This number must be a proper divisor of $32 - 2 = 30$. Removing 2, this leaves $3, 5, 6$,and 15 as possible values. Since we know that $\sqrt{\frac{|G|}{|L|}} + 1 \leq n$, we can eliminate 3 as a possible value. Substituting $n = 6$ into the equation $|G| = n(|H|) - (n-1)|L|$, we obtain $32 = 6(|H|) - (5)(2)$, from which it follows that $|H| = 7$, an impossibility in a group of order 32. It follows that such a cover must either involved 5 subgroups of order 8, or 15 subgroups of order 4.

The following groups of order 32 have been found in GAP to have equal C_2 intersection covers:

1. $C_4 \times C_2 \times C_2 \times C_2$
2. $C_2 \times C_2 \times D_4$
3. $(C_2 \times D_4) \ltimes C_2$
4. $(C_2 \times Q_8) \times C_2$
5. $C_2 \times ((C_4 \times C_2) : C_2)$
6. $(C_2 \times D_4) \ltimes C_2$

For each group, `GAP` found at least one cover by 5 subgroups of order 8 and at least one cover by 15 subgroups of order 4. Multiple covers by subgroups of order 8 were found for several instances.

Again, the groups in this list are expressed as the direct product or semi-direct products of C_2 with groups of order 16 which have equal C_2-intersection covers.

4.1.4 Groups of order 64

According to `GAP`, there are 267 groups of order 64. First, we need to determine the number and order of subgroups which would be involved in an equal C_2-intersection cover of such a group. Since n must be a proper divisor of 62 (64 - 2), n is equal to either 2 or 31. We know that $n = 2$ is impossible, so n must be equal to 31. Substituting this back into the equation $64 = 31(H) - 30(2)$, we see that the subgroups in an equal L-intersection cover must have order 4.

This particular collection illustrates some of the downsides of the brute-force approach. Many groups have substantially more than 31 subgroups of order 4, making it impossible to enumerate all the combinations. Even for a group such as $C_4 \times C_4 \times C_4$ which has 35 subgroups of order 4, this brute force approach would require GAP to calculate $\binom{35}{31} = 52360$ different potential covers. Therefore, we will need to use theory in conjunction with GAP to attack this and other cases.

Using GAP, we can disqualify the groups which do not have at least 31 subgroups of order 4. Since every element of G must be contained in a subgroup of order 4, we can also eliminate any group with exponent greater than 4. This leaves just 97 of the original 267 groups as candidates.

We can work around this problem by noting that if G has an equal C_2-intersection cover by 31 groups of order 4, there must be an element of order 2 which appears in 31 distinct 4-subgroups of G. Then we can use GAP to determine the elements in each subgroup and count the number of subgroups in which a particular element is contained.

We will use the following command to have GAP count how many times each element is contained in a subgroup of order k.

```
Collected(Flat(List(kSubgroups(G,k),
          x -> Elements(x)));
```

In the above command,

```
List(kSubgroups(G,k),x -> Elements(x)
```

returns a list of lists. The entries are lists corresponding the elements of every subgroup of a given order. The

```
Flat()
```

command concatenates this into a single list where the multiplicity of each group element corresponds to the number of subgroups of order k in which it appears. Finally,

```
Collected()
```

counts how many time each element appears in the flattened list; these counts tell us how many subgroups contained each group element.

The identity will appear in every subgroup. We modify the above command to ignore it by filtering the original list to only include elements with order greater than 1.

```
Collected(Flat(Union(List(kSubgroups(G,k),
          x -> Filtered(Elements(x),
          x -> Order(x) > 1)));
```

Finally, we can use the

```
Maximum
```

command in conjunction with the above command to determine the largest number of such subgroups in which any particular element is contained. Groups for which this number is less than the required value of n can be disqualified during a particular search. If G has an element of order 2 contained in exactly 31 subgroups of order 4, G must have an equal C_2-intersection cover. The proof of a more general version of this fact will be supplied in the next section.

Using `GAP` and our more refined approach to brute force, we can determine that the following groups have non-identity elements which appear in exactly 31 subgroups of order 4.

- $C_4 \times C_2 \times C_2 \times C_2 \times C_2$
- $C_2 \times C_2 \times C_2 \times D_4$
- $C_2 \times C_2 \times C_2 \times Q_8$
- $C_2 \times C_2 \times ((C_4 \times C_2) \ltimes C_2)$
- $C_2 \times ((C_2 \times D_4) \ltimes C_2)$
- $C_2 \times ((C_2 \times Q_8) \ltimes C_2)$

- $(C_2 \times ((C_4 \times C_2) \ltimes C_2)) \ltimes C_2$
- $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$

Hence, each group has an equal C_2-intersection cover. Again, each of these groups is the direct product or semi-direct product of C_2 with a group of order 32 which possesses an equal C_2-intersection cover.

4.1.5 Groups of Order 128

`GAP` tells us that there are 2328 groups of order 128, so we will restrict to our attention only to determining whether certain groups of this size have an equal C_2-intersection cover.

Supposing that G is a group of order 128 which has an equal C_2-intersection cover, we first wish to determine the size and number of subgroups in the cover. Since n must be a proper divisor of $128 - 2 = 126$ which is not equal to 2, we can say that n must be an element of $\{3, 6, 7, 9, 14, 18, 21, 42, 63\}$. Further, we know that $n \geq \sqrt{\frac{|G|}{|L|}} + 1$; hence, $n \geq 9$ and it follows that we can eliminate 3,6, and 7 from consideration.

Also, since n must be odd, we can eliminate $14, 18$, and 42 as choices of n. If $n = 9$, then $128 = 9|H| - (8)(2) = 9|H| - 16$, hence $144 = 9|H|$. Since 9 divides 144, this means that 9 subgroups of order 16 is one possibility. If $n = 21$, $21|H| = 168$, which implies that $|H| = 8$. If $n = 63$, $63|H| = 128 + 124 = 252$, whence $|H| = 4$. Thus G is covered by 9 subgroups of order 16, 21 subgroups of order 8, or 63 subgroups of order 4. Again, the number of subgroups of these orders makes exhaustive enumeration of the possible covers prohibitive at this time. We instead will focus on groups which fit the pattern previously established.

Based on what we have seen so far, we would expect groups isomorphic to one of

1. $C_4 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$
2. $C_2 \times C_2 \times C_2 \times C_2 \times D_4$
3. $C_2 \times C_2 \times C_2 \times C_2 \times Q_8$
4. $C_2 \times C_2 \times C_2 \times ((C_4 \times C_2) \ltimes C_2)$

5. $C_2 \times C_2 \times ((C_2 \times D_4) \ltimes C_2)$

6. $C_2 \times C_2 \times ((C_2 \times Q_8) \ltimes C_2)$

7. $C_2 \times (C_2 \times ((C_4 \times C_2) \ltimes C_2)) \ltimes C_2$, and

8. $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$

to have equal C_2-intersection covers. Based on a result previously proven, we know that $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$ has such a cover. One way to demonstrate that the other groups have such a cover is to find an element of order 2 which appears in 63 subgroups of order 4. Using `GAP`, we are able to find that each of the groups in the above list has such an element, and hence has an equal C_2-intersection cover.

4.2 Conjectures and Questions

In the previous section, we used `GAP` to enumerate all the 2-groups of order 128 or less which had equal C_2-intersection covers. For $n > 3$, the groups of 2^n which we encountered could be described as the direct or semi-direct product of a group isomorphic to C_2 and a group of order 2^{n-1}. This leads us to the following conjecture.

Conjecture 4.1. Let $k \geq 3$ and G be a group of order 2^k with an equal C_2-intersection cover. Then $G \times C_2$ is a group of order 2^{k+1} with an equal C_2-intersection cover.

The approach used to find the cover in this instance suggests a general test to determine when a 2-group has an equal C_2-intersection cover.

Proposition 4.2. *Let G be a group such that $|G| = 2^n$ for $n > 2$. If G has an element of order 2 which appears as an element in $2^{n-1} - 1$ subgroups of order 4, G has an equal C_2-intersection cover.*

Proof. Suppose that G is a finite group such that $|G| = 2^n$ for some $n > 2$. Suppose further that G has $2^{n-1} - 1$ subgroups of order 4, and that g is an element which appears in each of these subgroups. By assumption, these subgroups are of equal order. The pairwise intersection of distinct subgroups must contain only the identity and g, hence the pairwise intersection is a cyclic group of order 2. Finally, we must show that the union of these elements is equal to the order of the group.

The union of these subgroups must be a subset of G. We have that

$$\begin{aligned}(2^{n-1}-1)4-(2^{n-1}-2)2 &= 2^{n+1}-4-2^n+4 \quad = 2^{n+1}-2^n \\ &= 2^n(2-1) \\ &= 2^n\end{aligned}$$

Since $\bigcup_{i=1}^{n^{2-1}-1} H_i$ is a subset with 2^n elements in a group with 2^n elements, it must be equal to the group. This completes the proof. □

The above result depends on unique arithmetic properties of the number 2. We have been unable to find a generalization for an arbitrary prime p.

In this instance, however, we can provide an explicit description of the subgroups $\{H_i\}_{i=1}^{2n-1-1}$. If x is an element of order 2 contained in each of the H_i, then $L = \langle x \rangle$ has 2^{n-1} cosets, and $2^{n-1}-1$ cosets of the form aL for some $a \in G - L$. Each $H_i = aL \cup L = \langle x, a \rangle$ for some $a \in G - L$. This means that there is a one-to-one mapping from the nontrivial cosets of L to the subgroups of order 4 in the L-intersection cover.

In each of the cases we have demonstrated so far, every group which has had an equal C_2-intersection by subgroups of order 8 has also had such a cover by subgroups of order 4. In this instance, each subgroup of order 4 is equal to $aL \cup L$ for some $a \in G - L$. This motivates the following question:

Question 4.3. *Let G be a group of order 2^k for $k > 3$. Suppose that G has an equal C_2-intersection cover by n noncyclic subgroups of order 8. Does G also have an equal C_2-intersection cover by $3n$ subgroups of order 4?*

We have reason to believe that the answer to this question is yes. Suppose that $G = \bigcup_{i=1}^{n}$, where $|H_i| = |H_j| = 8$ for $1 \leq i, j \leq n$ and $i \neq j$, and $H_i \cap H_j = L \cong C_2$. Since any noncyclic subgroup of order 8 can be written as the union of 3 subgroups of order 4, we could replace each H_i with the subgroups $H_{i,1} \cup H_{i,2} \cup H_{i,3}$, of order 4 to obtain an equal cover.

For any choice $1 \leq i, j \leq n$ and any choice of k_1 and k_2 in $\{1, 2, 3\}$, H_{i,k_1} and H_{j,k_2} must be distinct, since otherwise H_i and H_j would intersect in a subgroup of order 4 not isomorphic to C_2. To finish proving the result, it would remain only to show that $H_{j,k_1} \cap H_{i,k_2} = L$ for any $1 \leq k_1, k_2 \leq 3$. In particular, we would need to show that $H_{j,k_1} \cap H_{i,k_2}$ is nontrivial. However, we do not know whether or not this final statement is true.

Finding the answer to this question would determine whether or not to investigate the following more general case.

Question 4.4. *Let G be a group of order 2^n. If G has an equal C_2-intesection cover by subgroups of order 2^k, where $2 < k < n$, must G also have an equal C_2-intersection cover by subgroups of order 4?*

If this question has an affirmative answer, it would reduce the study of equal C_2-intersection covers in 2-groups to those groups whose subgroups of order 4 are in one-to-one correspondence with the cosets of some cyclic 2-subgroup L. On the other hand, a counterexample would lead one to wonder whether there are equal C_2-intersection covers by subgroups of size 2^k for any integer k.

Another area of interest would be the relationship between the pairwise intersection and total intersection of subgroups in a cover. At this time, we are uncertain whether having a cover by equal subgroups whose pairwise intersections are isomorphic is enough to determine that all pairwise intersections must also be equal as sets.

Question 4.5. *Are there cases of groups with covers by subgroups of equal order which have isomorphic but unequal pairwise intersections?*

As with many other questions, the search for such a cover could be facilitated by using `GAP` to search for such examples.

The results given in this work are a first step towards investigating a new class of group covers, which are interesting in their own right and may be able to provide insights into open questions regarding other covers of finite groups. The examples which we present suggest several questions regarding the structure of 2-groups with a specified pairwise intersection isomorphic to C_2. The generalization to other groups and other isomorphism classes of intersection could prove very fruitful for future research.